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ON AN ALTERNATIVE FOR PURSUIT-EVASION GAMES IN AN INFINITE TIME INTERVAL*

A.A. AZAMOV

The structure of phase space of differential pursuit-evasion games is studied for the case when the evader is subjected to information discrimination with an advance by $\delta > 0$, $\delta = \mathrm{const}$. The method of transfinite iteration of the Pshenichnyi operator is used to establish an alternative for differential pursuit-evasion games in an infinite time interval.

1. Definition of the basic concepts. Consider a differential pursuit-evasion game in the phase space $\mathbf{R}^d$ with the equation of motion

$$z^{\cdot} = f(z, u, v),\ u \in P \subset \mathbf{R}^p,\ v \in Q \subset \mathbf{R}^q \tag{1.1}$$

and the terminal set $M \subset \mathbf{R}^d$. The aim of the pursuer who controls the parameter u is to bring the phase point from its initial position to the set N; the aim of the evader who controls the parameter v, is the opposite. At every instant of time $t \geqslant 0$ the pursuer has access to up to date information $z(t)$ and $v(s), t \leqslant s \leqslant t + \delta$ (everywhere in $\delta > 0$). Following /1/, we assume that the number δ is chosen by the evader at the start of the motion, and does not change during the motion (comp. with /2/).

The type of informability described above is realized below by separating specific classes of the strategies $\mathbf{P}, \mathbf{Q}$ of the pursuer and evader respectively (for brevity we will call such a pair of strategies the game). Here every triad $\xi \in \mathbf{R}^d$, $U \in \mathbf{P}$, $V \in \mathbf{Q}$ generates a unique trajectory $z(t; \xi, U, V)$, $t \geqslant 0$.

Let I denote either a segment of the form $[0, \tau]$, or a semi-axis $[0, \infty)$. By definition, a pursuit originating at the initial point ξ can be successfully completed in the interval I, if a strategy $U \in \mathbf{P}$ exists such that an inclusion $z(t; \xi, U, V) \in M$ occurs at some $t \in I$ whatever $V \in \mathbf{Q}$ is. Similarly, escape from the point ξ is possible in the interval I, if a strategy $V \in \mathbf{Q}$ exists such that we have $z(t; \xi, U, V) \overline{\in} M$ for any $t \in I$, whatever $U \in \mathbf{P}$ is.

Let I^+ (or I^-) denote the set of all points from which the pursuit (or possibly escape)

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The validity of the alternative has been established, up to now, for a wide class of games in a finite time interval /2-9/. In /3/ it was shown that the alternative occurs on the semi-axis $[0, \infty)$, provided that the condition of uniformity of evasion holds. An example is given in /3/, Sect.60, showing that in the general case the alternative need not occur on $[0,\infty)$ for the game (1.1) with positional strategies. In /2/ the author describes an analogous example for the case of the ε-strategies. Below we study the qualitative construction of the phase space and prove theorems on the alternative in an infinite time interval when the players use the strategies from the classes $\mathbf{P}, \mathbf{Q}$.

We denote the set of all measurable functions $a(\cdot) : [\alpha, \beta] \to A$ where $A \subset \mathbf{R}^r$), by $A\,[\alpha, \beta]$. In what follows, we shall assume that condition K holds: the domains P, Q are bounded, and for any $\xi \in \mathbf{R}^d$, $u(\cdot) \in P\,[\alpha, \beta]$, $v(\cdot) \in Q\,[\alpha, \beta]$ the Cauchy problem

$$z^{\cdot} = f(z, u(t), v(t)),\ z(0) = \xi \tag{1.2}$$

has a unique Caratheodory solution defined in the segment $[\alpha, \beta]$. If $\alpha = 0$, then the solution will be denoted by $z(t; \xi, u(\cdot), v(\cdot))$.

Definition. We will call the strategy of the pursuer the pair U, consisting of a family of operators $\{u^\delta\}$ and the functional $\{\tau^\delta\}$. Every operator u^δ (the functional τ^δ) maps the set $\mathbf{R}^d \times Q\,[0, \delta]$ onto the set $P\,[0, \delta]$ (or, respectively, into the interval $(0, \delta]$). We will call the strategy of the pursued the pair $V = (\delta, v_\delta)$ consisting of a positive number δ and the mapping $v_\delta : \mathbf{R}^d \to Q\,[0, \delta]$. The set of all strategies of the pursuer (pursued) is denoted by $\mathbf{P}\,(Q)$.

If the initial point ζ is given, then the action of the strategies $U = (\{u^\delta\}, \{\tau^\delta\})$ and $V = (\delta, v_\delta)$ generates a definite trajectory $z(t) = z(t; \zeta, U, V)$ as follows. Let $v_0(\cdot) = v_\delta[\zeta]$, $u_0(\cdot) = u^\delta[\zeta, v_0(\cdot)]$ and $t_1 = \tau^\delta[\zeta, v_0(\cdot)]$. The function $z(t)$ in the time interval $[0, t_1]$ is a solution of problem (1.2), in which $u(t) = u_0(t), v(t) = v_0(t), \alpha = 0$ and $\xi = \zeta$. Let $z_1 = z(t_1)$, $v_1(\cdot) = v_\delta[z_1], u_1(\cdot) = u^\delta[z_1, v_1(\cdot)]$ and $t_2 = t_1 + \tau^\delta[z_1, v_1(\cdot)]$. The function $z(t)$ is continued in the segment $[t_1, t_2]$ as a solution of problem (1.2) in which $u(t) = u_1(t - t_1), v(t) = v_1(t - t_1), \alpha = t_1$ and $\xi = z_1$. Repeating the above arguments a denumerable numbers of times, we obtain an increasing sequence of numbers t_n and the function $z(t), t \in [0, t_\omega)$ where $t_\omega = \sup t_n$. At the same time we generate the measurable functions $u_*(\cdot) : [0, t_\omega) \to P$, $v_*(\cdot) : [0, t_\omega) \to Q$ which we will call the realizations. For example, $u_*(t) = u_0(t)$ when $t \in [0, t_1), u_*(t) = u_1(t - t_1)$ when $t \in [t_1, t_2)$ etc.

If $t_\omega = \infty$, then the construction of trajectories ceases. In the general case we may find that $t_\omega < \infty$. In this case we must resort to transfinite induction /10/ in order to prolong the trajectory over the whole semi-axis $[0, \infty)$.

Let λ by any ordinal number and let the sequence t_μ be defined for all $\mu < \lambda$. If $t_* = \sup t_\mu = \infty$, then the construction is completed. Let $t_* < \infty$. If λ follows some serial number ν, then we assume that $\xi = z(t_\nu)$, $v_\nu(\cdot) = v_\delta[\xi]$, $u_\nu(\cdot) = u^\delta[\xi, v_\nu(\cdot)]$ and $t_\lambda = t_\nu + \tau^\delta[\xi, v_\nu(\cdot)]$ and the trajectory is continued onto the segment $[t_\nu, t_\lambda]$ as a solution of problem (1.2) in which $u(t) = u_\nu(t - t_\nu), v(t) = v_\nu(t - t_\nu), \alpha = t_\nu$. If the number λ has no preceding number, then we must write $t_\lambda = t_*$, continue the realization from the interval $[0, t_\lambda]$ to the segment $[0, t_\lambda]$ and put $z(t_\lambda) = \lim z(t)$ (the limit as $t \to t_\lambda - 0$, exists by virtue of the condition K).

Since the intervals $(t_\mu, t_{\mu+1})$ are non-empty and pairwise non-intersecting, it follows that $t_\lambda = \infty$ for some limiting serial number exceeding the first non-denumerable ordinal number. Thus we have constructed the trajectory and its uniqueness is obvious.

As was noted by Ushakov, the approach adopted here allows the pursued to terminate, at the instant $t_{\lambda+1} \in (t_\lambda, t_\lambda + \delta]$, the control $v_\lambda(t - t_\lambda), t_\lambda \leqslant t \leqslant t_{\lambda+1}$ chosen by him at $t = t_\lambda$, and to switch over to a new control $v_{\lambda+1}(\cdot)$. In the problem of escape this condition is natural. As far as the problem of pursuit is concerned, we can assume without loss of generality that the controls of the pursued are programmed.

2. Qualitative structure of the phase space.

Let the operator T_δ /2/ place, in correspondence, every subset $A \subset \mathbf{R}^d$ with the set

$$T_\delta(A) = \{z \in \mathbf{R}^d \mid \forall v(\cdot), \exists u(\cdot), \exists t \in [0, \delta] : z(t) \in A\}$$

where $u(\cdot)$ traverses $P\,[0, \delta]$, $v(\cdot)$ traverses $Q\,[0, \delta]$, $z(t) = z(t; z, u(\cdot), v(\cdot))$. Using the process of induction over λ, we introduce the transfinite sequence of subsets $M_\delta^\lambda \subset \mathbf{R}^d$, namely $M_\delta^0 = M$, $M_\delta^\lambda = T_\delta(M_\delta^\mu)$ for the serial number λ following μ and $M_\delta^\lambda = U\{M_\delta^\mu \mid \mu < \lambda\}$ for the limiting λ.

Proposition 1. Let λ be an ordinal number whose power exceeds the power of the continuum. Then $M_\delta^{\lambda+1} = M_\delta^\lambda$, i.e. $T_\delta(M_\delta^\lambda) = M_\delta^\lambda$.

Proof. Let $M_\delta^{\lambda+1} \neq M_\delta^\lambda$. Then $M_\delta^{\mu+1} \setminus M_\delta^\mu \neq \varnothing$ for all $\mu \leqslant \lambda$. The power of the set Λ of all isolated ordinal numbers smaller than λ also exceeds the power of the continuum. Let us place the point $a(\mu) \in M_\delta^{\mu+1} \setminus M_\delta^\mu$ in correspondence with every $\mu \in \Lambda$. Since the sets $M_\delta^{\mu+1} \setminus M_\delta^\mu$ do not pairwise intersect, it follows that the function $a(\cdot): \Lambda \to R^d$ is in 1:1 correspondence and this contradicts the continuous character of R^d.

We denote by $\sigma(\delta)$ the smallest ordinal number λ for which $M_\delta^{\lambda+1} = M_\delta^\lambda$. Several zones separate out naturally in phase space. Let $\tau > 0$, $\varepsilon = \tau/n$, where n is a positive integer.

Proposition 2. A pursuit from the points of the set $Z^\tau = \bigcap_n M_\varepsilon^n$ can be completed within the time interval $[0, \tau]$.

The strategy U^* ensuring the completion of the pursuit from the initial point $\xi \in Z^\tau$ is constructed below in the proof of Proposition 3. From Lemma 1 of /11/ it follows that the set Z^τ is the same as $\tilde{T}_\tau(M)$ /2/, provided that M is closed.

Corollary. A pursuit can be completed within a finite time interval from every point of the zone $Z^\infty = \bigcup_{\tau>0} Z^\tau$.

Let ω be a natural, ordinal-type series

$$Z^\delta = M_\delta^{\sigma(\delta)}, \quad Z^\omega = \bigcap_\delta M_\delta^\omega, \quad Z^+ = \bigcap_\delta Z^\delta$$

Proposition 3. A pursuit can be completed in the time interval $[0, \infty)$ from the points of the zone Z^+.

Proof. Let δ be arbitrary. We have the decomposition

$$\mathbf{R}^d = M \cup (\mathbf{R}^d \setminus Z^\delta) \cup \bigcup_{\lambda \in \Lambda} (M_\delta^\lambda \setminus M_\delta^{\lambda+1})$$

of the phase space into the pairwise non-intersecting parts. If $z \in M \cup (\mathbf{R}^d \setminus Z^\delta)$, then the values of the mappings u^δ, τ^δ play no part here. Let $z \in M_\delta^\lambda \setminus M_\delta^{\lambda-1}$, $v(\cdot) \in Q\,[0, \delta]$. Then $z \in T_\delta(M_\delta^{\lambda-1})$ and $u_*(\cdot) \in P\,[0, \delta]$ and $\theta \in [0, \delta]$ exist such that $z(\theta; z, u_*(\cdot), v(\cdot)) \in M_\delta^{\lambda-1}$. Since $z \,\overline{\in}\, M_\delta^{\lambda-1}$, we have $\theta > 0$. The relations $u^\delta[z, v(\cdot)] = u_*(\cdot)$, $\tau^\delta[z, v(\cdot)] = \theta$ determine the strategy $U^* \in \mathbf{P}$.

Let $\xi \in Z^+ \setminus M$, and let the pursuer apply the strategy U^* and the pursued an arbitrary strategy $V \in \mathbf{Q}$, $z(t)$ in the corresponding trajectory. Then $\xi \in Z^\delta$ where δ is the first component of V. Let λ_0 denote the smallest of the ordinal numbers λ for which $\xi \in M_\delta^\lambda$. Clearly, $\lambda_0 \in \Lambda$. The method of constructing the strategy U^* implies that $z(t_1) \in M_\delta^{\lambda_0-1}$.

Let λ_1 denote the smallest of λ for which $z(t_1) \in M_\delta^\lambda$. If $\lambda_1 = 0$, then the pursuit is completed, otherwise λ_1 will also have a number preceding it and we will have the inclusion $z(t_2) \in M_\delta^{\lambda_1-1}$. Repeating the above arguments we can construct a decreasing sequence of numbers λ_n for which $z(t_n) \in M_\delta^{\lambda_n}$. Since any sequence of ordinal numbers is fully ordered, it follows that $\lambda_n = 0$ for some n. Then $z(t_n) \in M$ and Proposition 3 is proved.

Let us now turn to the zone Z^ω. Clearly, $Z^\infty \subset Z^\omega \subset Z^+$. Examples given in Sect.4 show that situations where $Z^\omega \neq Z^+$, $Z^\omega \neq Z^\infty$ are possible. The following assertion characterizes the zone Z^ω. Let $\mathbf{Q}_\delta$ be a set of all strategies of the pursued whose first component is not less than δ.

Proposition 4. Let $\xi \in Z^\omega$. Then a pursuit can be completed from the point ξ in the interval $[0, \infty)$ in the game $(\mathbf{P}, \mathbf{Q})$, and in some finite time interval in the game $(\mathbf{P}, \mathbf{Q}_\delta)$, for any δ.

Thus a guaranteed finite time of pursuit exists for the points $\xi \in Z^\infty$. In the case of the points $\xi \in Z^\omega$ we can guarantee a finite time of pursuit, provided that the pursued employs, generally speaking, a strategy from the class $\mathbf{Q}_\delta$ and δ is known to the pursuer. In the case when $\xi \in Z^\omega \setminus Z^\infty$, the pursued will be able to prevent the completion of the pursuit in a finite period of time, by reducing the component δ of his strategy, whereas he can achieve it for the initial points $\xi \in Z^+ \setminus Z^\omega$, also for a fixed δ, by a suitable choice of the second component v_δ.

3. The alternative in the infinite time interval.

Let $\mathbf{P}^\omega$ be the set of all strategies $U \in \mathbf{P}$ possessing the following property: for any $\xi \in \mathbf{R}^d$ and $V \in \mathbf{Q}$ the sequence t_λ of the switchovers of the trajectory $z(t; \xi, U, V)$ is an ordinary (not transfinite) sequence, i.e. $t_\omega = \infty$. The class $\mathbf{P}^\omega$ is non-empty. It contains e.g. the strategy U^* constructed in the proof of Proposition 3.

Proposition 5. Let $Z^- = \mathbf{R}^d \setminus Z^+$. Then an escape from the points of the zone Z^- is possible in the game $(\mathbf{P}^\omega, \mathbf{Q})$ within the time interval $[0, \infty)$.

Proof. If $\xi \in Z^-$, then $\xi \,\overline{\in}\, Z^\delta$ for some δ. The number δ is used as the first component of the strategy V_*. The component v_δ is chosen as follows. If $z \,\overline{\in}\, Z^\delta$, then $z \,\overline{\in}\, T_\delta(Z^\delta)$ by virtue of the choice of $\sigma(\delta)$. Therefore $v_*(\cdot) \in Q[0, \delta]$ exists such, that $z(t; z, u(\cdot), v_*(\cdot)) \overline{\in} Z^\delta$ irrespective of what $u(\cdot) \in P[0, \delta]$ and $t \in [0, \delta]$ are. Let $v_\delta[z] = v_*(\cdot)$. If $z \in Z^\delta$, then the value of $v_\delta[z]$ plays no part here.

The strategy V_* ensures the escape from the initial point ξ. Indeed, let $U \in \mathbf{P}^\omega$, $z(t) = z(t; \xi, U, V_*)$ and t_n be the corresponding sequence of the switchover instances. Since $\xi \overline{\in} Z^\delta$, it follows that $z(t) \overline{\in} Z^\delta$ for all $t \in [0, \delta]$, and in particular when $t \in [0, t_1]$. If we put $z(t_1) = z_1$, then $z(t - t_1; z_1, U, V_*) = z(t; \xi, U, V_*) \overline{\in} Z^\delta$ for $t \in [t_1, t_2]$, etc. Since $M \subset Z^\delta$, this proves proposition 5.

From Propositions 3 and 5 we have

Theorem 1. Let the pursuer in the differential game (1.1) satisfying the condition *K*, adopt the strategy of class $\mathbf{P}^\omega$ and the pursued the strategy of class $\mathbf{Q}$. Then we have an alternative in the time interval $[0, \infty)$.

Henceforth we will assume in Sect.3 that a) the domains of controls $\mathbf{P}$ and $\mathbf{Q}$ are compact; b) the function f is continuous and satisfies locally the Lipschitz condition in the variable z; c) for any $\xi \in \mathbf{R}^d$, $u(\cdot) \in P[\alpha, \beta]$, $v(\cdot) \in Q[\alpha, \beta]$ problem (1.2) has a solution which can be continued to the segment $[\alpha, \beta]$; d) the vectogram $f(z, u, Q)$ is convex for any $z \in \mathbf{R}^d$, $u \in P$; e) the terminal set M is open. We prove in the same manner as Lemma 1 of /12/, that if $u(\cdot) \in P[0, \delta]$ and the subset $A \subset \mathbf{R}^d$ is compact, then the set $\Phi = \{z(\cdot; \xi, u(\cdot), v(\cdot)) \mid \xi \in A,\ v(\cdot) \in Q[0, \delta]\}$ is bounded in the norm of the space $\mathbf{C}[0, \delta]$. The compactness of Φ in the topology $\mathbf{C}[0, \delta]$ easily follows.

Proposition 6. The operator T_δ transforms open sets into open sets.
The proof follows that of property 3 in /2/.

Corollary. All sets M_δ^λ, and in particular $M_\delta^\omega, Z^\delta$, are open.

Proposition 7. $\sigma(\delta) \leqslant \omega$ for every δ.

Proof. Let $\xi \overline{\in} M_\delta^\omega$, so that $\xi \overline{\in} M_\delta^n$ at any $n \geqslant 1$. Therefore $v_n(\cdot) \in Q[0, \delta]$ exists such that $z_n(t) = z(t; \xi, u(\cdot), v_n(\cdot)) \overline{\in} M_\delta^{n-1}$ for any $u(\cdot) \in P[0, \delta]$ and $t \in [0, \delta]$. Since the sequence M_δ^n increases with respect to the inclusion, it follows that

$$z_n(t) \overline{\in} M_\delta^l \text{ when } n > l \tag{3.1}$$

We can separate from the sequence $z_n(\cdot)$ a subsequence, which converges in the norm of $\mathbf{C}[0, \delta]$ to the trajectory $z_*(t) = z(t; \xi, u(\cdot), v_*(\cdot))$ corresponding to some control $v_*(\cdot) \in Q[0, \delta]$. Since M_δ^l is open, the relations (3.1) imply that $z_*(t) \overline{\in} M_\delta^l$ for $t \in [0, \delta]$. Therefore $z_*(t) \overline{\in} M_\delta^\omega$. From this we have $\xi \overline{\in} T_\delta(M_\delta^\omega)$, so that $\mathbf{R}^d \setminus M_\delta^\omega \subset \mathbf{R}^d \setminus T_\delta(M_\delta^\omega)$. This inclusion is equivalent to the relation $M_\delta^{\omega+1} = M_\delta^\omega$.

Proposition 8. Escape is possible from the points of the zone Z^- in the game $(\mathbf{P}, \mathbf{Q})$ in the time interval $[0, \infty)$.

This is proved in exactly the same manner as Proposition 5.

Theorem 2. Let the pursuer in the differential game satisfying the conditions a)-e), apply the strategy from class $\mathbf{P}$, and the pursued the strategy from class $\mathbf{Q}$. Then an alternative exists in the time interval $[0, \infty)$.

4. Examples. 1°. Let five points x_i $(i = 0, 1, \ldots, 4)$ move in the plane $\mathbf{R}^2$ according to the equations

$$x_i^{\cdot} = u_i, \quad x_i \in \mathbf{R}^2 \tag{4.1}$$

where $u_4 = 0$ and $|u_i| \leqslant 1$ when $i = 0, 1, 2, 3$. The point x_0 is controlled by the pursued, and the points x_1, x_2, x_3, x_4 by the pursuer (here x_4 is fixed). The pursuit is completed when $x_i = x_0$ for at least one $i = 1, 2, 3, 4$.

The game is linear and has the phase space R^{10} and a closed terminal set.

Let $z_0 = (x_0^0, x_1^0, \ldots, x_4^0)$ and let A denote the interior (in the topology of $\mathbf{R}^2$) of the convex envelope of the points x_1^0, x_2^0, x_3^0.

The phase space has the following construction. The zone Z^∞ consists of the points z_0 for which $x_0^0 \in A$. The zone Z^ω coincides with Z^∞ and the zone Z^+ consists of Z^∞ as well as those points z_0 for which $A \neq \varnothing$, x_0^0 lies at the boundary A, while the ray emerging from the point x_0^0 in the direction of $x_4^0 - x_0^0$, perpendicular to a side of the triangle A, lies outside it. Here $\sigma(\delta) \leqslant 2\omega$ for every δ.

2°. Let the differential game in the plane with coordinates x, y be described by the equations

$$x^{\cdot} = 1, \; y^{\cdot} = v, \; 0 \leqslant v \leqslant 1 \tag{4.2}$$

The terminal set is defined as follows. Let the subset $\Sigma \subset [0, \infty)$ be completely ordered with respect to the natural order of the numerical axis, and be of the ordinal type ρ, with $0 \in \Sigma$. Let $l(n, \alpha)$ be a segment of the straight line $y = -x + n + \alpha$ bounded by the hyperbola $y = (x\alpha^+ + \alpha)/(x + 1)$, $x \geqslant -1$ and the ray $y = \alpha, x \geqslant 0$ $(n \in \mathrm{N}, \alpha^+$ denotes the number following the element $\alpha)$ in Σ. We take, as the terminal set M, the union of all possible segments

$l(n, \alpha)$, $n \in \mathbf{N}$, $\alpha \in \Sigma$ and the region $xy \geqslant -1$, $x \geqslant 0$. The set M is closed.

The sequence M_δ^λ in the game (4.2) is independent of δ and increases up to $\omega\rho$, i.e. to the value of the product of the ordinal numbers ω and ρ. The zones Z^∞, Z^ω coincide here with the union of the set M and half-plane $y < 0$. If $b = \sup \Sigma \in \Sigma$, then $Z^+ = \{(x, y) \mid y \leqslant b\}$, otherwise $Z^+ = \{(x, y) \mid y < b\}$.

3°. The region $D \subset \mathbf{R}^2$ is given in polar coordinates by the inequalities $1{,}5 + \varphi/(1 + \varphi) < r < 1{,}5 + (2 + \varphi)/(1 + \varphi)$, $\varphi > 0$. The point p, controlled by the pursuer, moves along the abscissa with a velocity not exceeding 1. The point q controlled by the pursued, moves so that its radius vector r rotates with an angular velocity of 1 and varies according to the law $r^\cdot = v$, $|v| \leqslant 2$. The pursuit is completed when $p = q$ or $q \overline{\in} D$.

The differential game given here has phase space $\mathbf{R}^3$, $P = [-1, 1]$, $Q = [-2, 2]$. Let z_1 be the abscissa of the point p, (z_2, z_3) are the coordinates of the point q. Then the equation of motion has the form

$$z_1^\cdot = u$$
$$z_2^\cdot = z_2 v/r - z_3, \quad z_3^\cdot = z_3 v/r + z_2, \quad r \geqslant 1$$
$$z_2^\cdot = r(z_2 v - z_3), \quad z_3^\cdot = r(z_3 v + z_2), \quad r < 1$$

(when $r < 1$, the right-hand sides are written out so that conditions a)-e) of Sect.3 hold, and the terminal set is

$$M = \{(z_1, z_2, z_3) \mid (z_1 = z_2, z_3 = 0) \quad \text{or} \quad (z_2, z_3) \overline{\in} D\}.$$

In this game $Z^\infty \neq Z^\omega$, e.g. $(0; 2; 0{,}1) \in Z^\omega \setminus Z^\infty$.

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